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# FUNCTIONAL DIFFERENCE EQUATIONS AND AN EPIDEMIC MODEL+

by

Lawrence Turyn
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912



For presentation at the International Conference on
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Lawrence Turyn

Lefschetz Center for Dynamical Systems
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## ABSTRACT

We consider an epidemic model of the form $S \to I \to S$ with history on $(-\infty,0]$. The well-known threshold phenomenon is discussed in terms of the stability of a functional difference equation, also known as the translation-invariant renewal equation. Since the difference equation has infinite delay, the work of other authors on finite-delay problems is extended. Also, epidemic models with spatial effects are discussed by extension of the results to difference equations in a Banach space.

## FUNCTIONAL DIFFERENCE EQUATIONS AND AN EPIDEMIC MODEL

Recently there has been interest in a model for the evolution of a disease given by the equation

$$(1) \qquad \frac{d}{dt} S(t) = S(t) \int_{-\infty}^{0} B(\theta) \frac{d}{dt} S(t+\theta) d\theta,$$

where $S(t)$ is the density of susceptible individuals and the right hand side allows for infectious contacts at the present time due to the past history of infections. Equation (1) is an $S \to I$ model, that is, it allows only for susceptibles to become infected. Diekmann [2,3] has considered this model and also has allowed spatial effects. In an $S \to I \to S$ model, where one allows for recovery (without immunity) of infecteds, one would add a term $\int_{-\infty}^{0} C(\theta) \frac{d}{dt} S(t+\theta) d\theta$ to the right-hand side of equation (1).

The "threshold phenomenon" of Kermack and McKendrick is well-known in mathematical epidemiology; see, for example, Bailey [1] or Hoppensteadt [11]. For equation (1) the appropriate initial data consists of the value $k = S_0 = S(0)$ and the initial history $\phi(\theta) = \frac{d}{d\theta} S(\theta)$, $\theta \in (-\infty,0]$. The epidemiological model is realistic only if $S_0 > 0$, $\phi(\theta) \leq 0$ and $B(\theta) \geq 0$ for $\theta \in (-\infty,0]$, and $B(\cdot)\phi(\cdot) \not\equiv 0$. With these assumptions there always exists $\lim_{t\to\infty} S(t;\phi,S_0) \overset{\text{defn}}{=} S_\infty(\phi;S_0)$. The threshold phenomenon can be stated as: There exists an $S^*$ such that

(a) For fixed $S_0 > S^*$, $\lim_{\phi\to 0} S_\infty(\phi;S_0) > S_0$.

(b) For fixed $S_0 < S^*$, $\lim_{\phi\to 0} S_\infty(\phi;S_0) = S_0$.

Here, $\lim_{\phi \to 0}$ stands for the limit in the function space where the initial history $\phi$ comes from.

The equation (1) can be transformed into an equivalent functional difference equation for $y(t) \stackrel{\text{defn}}{=} \frac{d}{dt} S(t)$ by noting that $S(t) = S_0 + \int_0^t y(\tau)d\tau$. Specifically, (1) is equivalent to

$$y(t) - k \int_{-\infty}^{0} B(\theta)y(t+\theta)d\theta = \left(\int_{-t}^{0} y(t+\theta)d\theta\right)\left(\int_{-\infty}^{0} B(\theta)y(t+\theta)d\theta\right), \tag{2}$$

where we have notated $k = S_0$ so as to distinguish $S_0$ as a parameter to be adjusted. We will show that the zero function $\phi(\theta) = 0$ for $\theta \in (-\infty, 0]$ is, for equation (2),

(a') uniformly asymptotically stable for $k < S^*$

(b') unstable for $k > S^*$

(c') $S^* = 1/\left(\int_{-\infty}^{0} B(\theta)d\theta\right)$,

and then we will show how the stability results (a'), (b') for equation (2) imply the threshold phenomenon (a), (b) for equation (1). Thus, we will show that the threshold phenomenon is a particular case of stability results for functional difference equations.

The greater part of the mathematical analysis of this paper will be the discussion of a particular class of functional difference equations which will be wide enough to obtain the result of the threshold pehnomenon. We will consider the equation

$$Dy_t = G(t, y_t) \tag{3}$$

where $y_t$ is a function of $\theta \in (-\infty,0]$ defined by $y_t(\theta) = y(t+\theta)$ and $D$ and $G(t,\cdot)$ are defined on function $\phi: (-\infty,0] \to \mathbb{R}^n$, with $D\phi = \phi(0) - \int_{-\infty}^{0} e^{\gamma\theta}A(\theta)\phi(\theta)d\theta$ so that $D$ is linear and of a restricted nature. Here, $A(\theta)$ is $\mathbb{R}^{n\times n}$ (real $n\times n$ matrix)-valued. The positive real number $\gamma$ is assumed to be fixed and $\int_{-\infty}^{0} |A(\theta)|d\theta < \infty$, where $|\cdot|$ is the Euclidean norm for $R^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$. By consideration of the spectral theory developed below it will hopefully become clear why the special form $e^{\gamma\theta}A(\theta)$, with $\int_{-\infty}^{0} |A(\theta)|d\theta < \infty$, is assumed for the kernel of the integral term in the linear operator $D$.

Since (3) has infinite delay some care must be taken in the assumptions made about the class of equations and class of solutions to be considered. Infinite delay in retarded functional differential equations has been considered by several authors, including Hale [5], Hale and Kato [7], and Naito [12]. We will look for solutions $y_t$ in the space

$$C_\gamma \overset{\text{defn}}{=} \{\phi: (-\infty,0] \to \mathbb{R}^n \mid \phi \text{ is continuous and there exists } \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)\}.$$

With $D$ restricted to the above special form it would not be difficult to work in the space $\mathbb{R}^n \times L_{p,\gamma}$, where

$$\mathbb{R}^n \times L_{p,\gamma}, \text{ where } L_{p,\gamma} = \{\phi \mid \int_{-\infty}^{0} e^{p\gamma\theta}|\phi(\theta)|^p d\theta < \infty\},$$

but we will not discuss this further. Also, it should be possible to develop a theory for the abstract spaces of Hale-Kato type, as in Hale and Kato [7] and Naito [12].

Note that for every $\phi \in C_\gamma$ we have (a) $\infty > |\phi|_\gamma \overset{\text{defn}}{=}$ $\overset{\text{defn}}{=} \sup_{\theta \leq 0} e^{\gamma\theta} |\phi(\theta)|$, and (b) the function $e^{\gamma \cdot}\phi(\cdot)$ is uniformly continuous for $\theta \in (-\infty, 0]$. $C_\gamma$ is a Banach space when given the norm $|\cdot|_\gamma$. Because

$$|D\phi| \leq \left(1 + \int_{-\infty}^{0} |A(\theta)| d\theta\right) |\phi|_\gamma,$$

$D: C_\gamma \to \mathbb{R}^n$ is a bounded linear operator.

For the general initial value problem (4) with initial data $y_0 = \phi \in C_\gamma$ we allow any $D: C_\gamma \to \mathbb{R}^n$ of the form

$$D\phi = \phi(0) - \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta) d\theta \quad \text{with} \quad \int_{-\infty}^{0} |A(\theta)| d\theta < \infty$$

and we allow any $G \in C^1(\mathbb{R}^+ \times C_\gamma; \mathbb{R}^n)$ for which the Frechet differential differential $D_\phi G(\cdot,\cdot)$ is bounded on all sets $\mathbb{R}^+ \times B$, $B$ any bounded subset of $C_\gamma$. For any initial data $\phi \in C_\gamma$ for which $D\phi = G(0,\phi)$ there exists a unique global solution $y_t$, i.e., $Dy_t = G(t, y_t)$ and $\lim_{t \to 0_+} |y_t - \phi|_\gamma = 0$, as can be proven using the contraction mapping theorem.

When $G \equiv 0$, so that (4) is linear, there is a unique global solution $y_t = y_t(\phi)$ for all initial data $\phi$ satisfying $D\phi = 0$. Defining $T_D(t)\phi = y_t$, we find that $T_D(t)$ is a semi-group of bounded linear operators on the Banach space $\mathcal{D} \overset{\text{defn}}{=} \{\phi \in C_\gamma : D\phi = 0\}$ with the $|\cdot|_\gamma$ norm. One can show that $T_D(t)$ is generated by the operator $A_D$ defined by $A_D\phi = \dot{\phi}$, $\cdot = \frac{d}{d\theta}$, on its domain $\mathscr{D}(A_D) = \{\phi \in \mathscr{B}: \dot{\phi} \in \mathscr{B}\}$.

The spectral properties of $T_D(t)$ can be found by examining the characteristic equation $0 = \det \Delta(\lambda) = \det\left(I - \int_{-\infty}^{0} e^{(\lambda+\gamma)\theta} A(\theta)\,d\theta\right)$, as is made rigorous by considering the generator $A_D$ and the particular nature of the equation $Dy_t = 0$. Following Naito [12], let us notate $\mathbb{C}_{-\gamma} = \{\lambda: \operatorname{Re} \lambda > -\gamma\}$. Given a linear operator B, we say that $\lambda$ is in (a) the <u>resolvent</u> of B if $(\lambda I-B)^{-1}$ is well-defined and bounded, (b) the <u>spectrum</u> of B if it is not in the resolvent. The <u>point spectrum</u> of B consists of those $\lambda$ for which $(\lambda I-B)$ has a non-trivial null-space.

<u>Theorem 1</u>: (i) The resolvent of $A_D$ is $\{\lambda \in \mathbb{C}_{-\gamma}: \det \Delta(\lambda) \neq 0\}$

(ii) The point spectrum of $A_D$ consists of $\{\lambda \in \mathbb{C}_{-\gamma}: \det \Delta(\lambda) = 0\}$, and if $\det \Delta(-\gamma) = 0$, also $\lambda = -\gamma$.

(iii) The spectrum of $A_D$ contains $\mathbb{C} \setminus \mathbb{C}_{-\gamma}$.

(iv) $A_D$ has compact resolvent, i.e. $(A_D-\lambda I)^{-1}$ is compact for all $\lambda$ for which it is a bounded linear operator on $C_\gamma$. As a consequence, the generalized eigenspace $\mathscr{M}(\lambda) \overset{\text{defn}}{=} \bigcup_{j=1} \mathscr{N}((A_D-\lambda I)^k)$ is of finite dimension, $\mathscr{N}$ denoting the null-space.

For (iv), use the compactification $[-\infty,0] = (-\infty,0] \cup \{-\infty\}$, the Arzela-Ascoli Theorem, and a well-known theorem on projections (found, for example, in Hille and Phillips [10, p. 182]).

Using the spectral properties of $A_D$ it is possible to obtain some, but not all, of the spectral properties of $T_D(t)$. From Hille and Phillips [10, p. 467] we have $P\sigma(T_D(t))\setminus\{0\} = \exp(P\sigma(A_D)\cdot t)$, where $P\sigma$ stands for the point spectrum. For the rest of the spectrum we must study the particular equation $Dy_t = 0$, rather than relying only on abstract results for semi-groups.

Define the difference operator $D_0\colon C_\gamma \to \mathbb{R}^n$ by $D_0\phi = \phi(0)$, and denote by $T_{D_0}(t)$ the semi-group for the difference equation $y(t) = D_0 y_t = 0$ defined on the Banach space $\mathscr{B}_0 \overset{\text{defn}}{=} \{\phi \in C_\gamma\colon D_0\phi = 0\}$ with the $|\cdot|_\gamma$ norm. As an aside, note that $|T_{D_0}(t)\phi|_\gamma \le e^{-\gamma t}|\phi|_\gamma$ for all $\phi \in \mathscr{B}_0$. Define the projection $\Psi_0\colon \mathscr{B} \to \mathscr{B}_0$ by $(\Psi_0\phi)(\cdot) = \phi(\cdot) - \phi(0)$.

<u>Lemma 2</u>: $T_D(t) = T_{D_0}(t)\Psi_0 + U(t)$ with $U(t)$ completely continuous on $\mathscr{B}$.

Let now $|\cdot|$ also stand for the operator norm on $L(\mathscr{B})$ = (the space of all bounded linear operators on $\mathscr{B}$). Using Lemma 2 and arguments involving the so-called "essential spectrum", one can prove, as was done by Hale [6, p. 285] and Henry [9, p. 117] for finite delay:

<u>Theorem 3</u>: Let $a_D = \max[-\gamma, \sup\{\mathrm{Re}\,\lambda\colon \det\Delta(\lambda) = 0\}]$. Then for all $\alpha > a_D$ there exists $K = K(\alpha)$ such that $|T_D(t)| \le Ke^{\alpha t}$ for all $t \ge 0$. One calls $a_D$ the <u>order</u> of the semi-group $T_D(t)$.

<u>Theorem 4</u>: For any $-\gamma \le \alpha \le \beta \le a_D$ the set $\Lambda = \{\lambda\colon \alpha < \mathrm{Re}\,\lambda < \beta,\ \det\Delta(\lambda) = 0\}$ has only finitely-many points.

For Theorem 4 a more specific reference is Hale [6, p. 309].

Using knowledge of the linear problem $Dy_t = 0$ we can discuss the nonlinear problem (3) by obtaining the variation of constants formula found below in Theorem 5. Let $X(t)$ be the fundamental matrix, i.e. the solution in $\mathbb{R}^{n\times n}$ of the equation $DX_t = I$ for $t \geq 0$ with initial data $X_0$ given by $X_0(0) = I$, $X_0(\theta) = 0$ for $\theta < 0$.

Theorem 5: The general solution $y_t \in C_\gamma$ of the inhomogeneous equation $Dy_t = h(t)$, $y_0 = \phi \in C_\gamma$ with $h(t)$ continuous for $t \geq 0$ and with $D\phi = h(0)$ is given by

$$x_t - X_0 h(t) = T_D(t)(\phi - X_0 h(0)) - \int_0^{t+} [d_s T_D(t-s)X_0]h(s). \tag{4}$$

Let us assume now that $G(t,0) = 0$, $D_\phi G(t,0) = 0$, and that $G(t,\cdot)$ depends weakly on the value of $\phi(0)$, specifically in the sense that

$$G(t,\phi \pm X_0 b) = G(t,\phi), \text{ for all } \phi \in C_\gamma \text{ and } b \in \mathbb{R}^n. \tag{5}$$

Define a new space $PC_\gamma = C_\gamma \oplus$ (the span of the columns of $X_0$) with norm $|\phi + X_0 b|_\gamma = |\phi|_\gamma + |b|_{\mathbb{R}^n}$, as in Hale and Martinez-Amores [8]. Using estimates on $X(\cdot)$ and the measures $d_s X(\cdot - s)$, as in Hale [6, p. 303], the space $PC_\gamma$, the variation of constants formula, and Gronwall's inequality, one can justify linearization using

Theorem 6: Assume $G \in C^1(\mathbb{R}^+ \times C_\gamma; \mathbb{R}^n)$, $G(t,0) = 0$, $D_\phi G(t,0) = 0$, and $D_\phi G(\cdot,\cdot)$ is bounded on all sets $\mathbb{R}^+ \times B$, $B$ bounded in $C_\gamma$, and that $G$ satisfies (5). Assume that $|G(t,\phi)| \leq Mg(|\phi|_\gamma)$ where $M$ is a positive real number and $g$ is continuously differentiable with $g(0) = 0 = g'(0)$. If $a_D < 0$ then $0 \in C_\gamma$ is uniformly asymptotically stable for equation (3).

Theorem 7 (Instability): Make the same assumptions on $G$ as in Theorem 6. If $a_D > 0$ then $0 \in C_\gamma$ is unstable for equation (3).

We have restricted ourselves to $D$ of the form

$$D\phi = \phi(0) - \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta)d\theta \quad \text{with} \quad \int_{-\infty}^{0} |A(\theta)|d\theta < \infty.$$

All of the above results can be achieved when one allows point delays, i.e. for $D\phi = \phi(0) - \sum_k e^{-\gamma r_k} A_k \phi(-r_k) - \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta)d\theta$ with $\sum_k |A_k| + \int_{-\infty}^{0} |A(\theta)|d\theta < \infty$, as long as one assumes $0 < r_1 < r_2 < \ldots$ (or, more generally, that $D$ is "atomic at zero"). We discuss the case with point delays in a paper currently in preparation.

Now we can show that the threshold phenomenon for equation (1) is equivalent to the question of the stability of equation (2). Since equation (2) is an example of equation (3) for which $G$ satisfies the assumptions of Theorem 6, the stability of (2) can be discussed by examining the stability of the linear difference operator $D(k)$ given by $D(k)\phi = \phi(0) - k\int_{-\infty}^{0} B(\theta)\phi(\theta)d\theta$, as long as we assume that $\int_{-\infty}^{0} e^{-\gamma\theta}|B(\theta)|d\theta < \infty$ for some constant $\gamma > 0$.

If $S^* = 1/\left(\int_{-\infty}^{0} B(\theta)d\theta\right)$ then it is easy to see that (a) $k > S^*$ implies $a_{D(k)} > 0$, and (b) $k < S^*$ implies $a_{D(k)} < 0$. Using $S(t) = S_0 + \int_{-t}^{0} v(t+\theta)d\theta$, where $v(t) = \frac{d}{dt} S(t)$ and $v$ solves (2), along with Theorems (6), (7), we can interpret the threshold phenomenon as a particular case of results in the stability theory of functional difference equations: For $k < S^*$, and thus $a_{D(k)} < 0$, it is clear that $S(\infty) = S_0 + O(|\phi|_\gamma)$, where $\phi(\theta) = v(\theta) = \frac{d}{d\theta} S(\theta)$ for $\theta(-\infty,0]$. For $k > S^*$, and thus $a_{D(k)} > 0$, the semi-group $T_D(t)$ has eigensolutions $e^{\lambda t}\phi^\lambda$, where $\phi^\lambda(\theta) = e^{\lambda\theta}$ is in $C_\gamma$ for some values of $\lambda$ with $\text{Re}\,\lambda > 0$. In fact, $\lambda = a_D$ is such a value! This latter property follows from the fact that $T_D(t)$ is a positive operator on the ordered Banach space $C_\gamma$, since $A(\theta) \geq 0$ $\theta \in (-\infty,0]$. Let $\hat{\phi} = \phi^{a_D}$, for notational convenience. For the epidemiological problem, the initial history $\phi$ satisfies $\phi(\theta) \leq 0$, $\theta \in (-\infty,0]$, and $\phi(\cdot) \not\equiv 0$, so that there is non-zero projection of $\phi$ onto the subspace $[\hat{\phi}]$ of $C_\gamma$ spanned by $\hat{\phi}$. From this, it follows that $\lim_{\phi\to 0} S_\infty(\phi;S_0) < S_0$ whenever $S_0 > S^*$, by using the general saddle-point theory, as in Hale [4, p. 157] or Henry's forthcoming book, Geometric Theory of Partial Differential Equations.

Diekmann [3] has allowed spatial effects in an $S \to I$ model to arrive at the equation $\frac{\partial}{\partial t} S(t,x) = S(t,x)\cdot\int_{-\infty}^{0}\int_{\Omega} B(\theta;x,\xi)S(t+\theta,\xi)d\xi d\theta$ in some region $\Omega \subset \mathbb{R}^m$. If $X$ is the ordered Banach space $C(\Omega)$ we can re-write this model as a functional difference equation in the ordered Banach space $C_\gamma = \{\phi\colon (-\infty,0] \to X \mid \phi$ continuous and there exists $\lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)\}$. If we restrict the linear operator

$D: C_\gamma \to X$ to the form $D\phi = \phi(0) - \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta)d\theta$, with the map $\phi \mapsto \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta)d\theta : C_\gamma \to X$ being completely continuous and order-preserving, then we get all of the above results, especially Theorems 6 and 7. From this we can get threshold results for the spatial model that are sharper than those of Diekmann [3], without the monotonicity assumption of Thieme [14, p. 103, p. 94].

For the $S \to I \to S$ model, where infecteds can recover, it should be possible to prove the existence of periodic solutions, via Hopf bifurcation, when the spectrum of the difference operator $D(k)$ given by

$$D(k)\phi = \phi(0) - k \int_{-\infty}^{0} e^{\gamma\theta} A(\theta)\phi(\theta)d\theta + \int_{-\infty}^{0} e^{\gamma\theta} C(\theta)\phi(\theta)d\theta$$

depends appropriately on the parameter $k$. Smith [13], among others, has investigated the existence of periodic solutions above the threshold.

## BIBLIOGRAPHY

[1] Bailey, Norman T.J., The Mathematical Theory of Infectious Diseases and Its Applications (New York: Hafner Press, 1975).

[2] Diekmann, O., "Limiting Behaviour in an Epidemic Model", Nonlinear Analysis, Theory, Methods and Applications, v. 1 (1977), pp. 459-470.

[3] Diekmann, O., "Thresholds and Travelling Waves for the Geographical Spread of Infection", J. Math. Biology, v. 6 (1978), pp. 109-130.

[4] Hale, Jack K., Ordinary Differential Equations (New York: Wiley-Interscience, 1969).

[5] Hale, Jack K., "Functional Differential Equations with Infinite Delays", JMAA, v. 48(1974), pp. 276-283.

[6] Hale, Jack K., Theory of Functional Differential Equations (New York: Springer-Verlag, 1977).

[7] Hale, Jack K. and Junji Kato, "Phase Space for Retarded Equations with Infinite Delay", Funkcialaj Ekvacioj, 21(1978)

[8] Hale, Jack K. and Pedro Martinez-Amores, "Stability in Neutral Equations", NA,T,M&A v. 1(1977), pp. 161-173.

[9] Henry, Daniel, "Linear Autonomous Neutral Functional Differential Equations", J.D.E., v. 15(1974), pp. 106-128.

[10] Hille, Einar and Ralph S. Phillips, Functional Analysis and Semi-groups (Providence, Rhode Island: American Mathematical Society Colloquium Publications, 1957).

[11] Hoppensteadt, Frank, Mathematical Theories of Populations: Demographics, Genetics, and Epidemics (Philadelphia: Society for Industrial and Applied Mathematics, 1975).

[12] Naito, Toshiki, "On Autonomous Linear Functional Differential Equations with Infinite Retardations", J.D.E., 21(1976), pp. 297-315.

[13] Smith, H.L., "On Periodic Solutions of a Delay Integral Equation Modelling Epidemics", J. Math. Bio., 4(1977), pp. 69-80.

[14] Thieme, Horst R., "Asymptotic Estimates of the Solutions of Nonlinear Integral Equations and Asymptotic Speeds for the Spread of Populations", J. Reine Angewandte, 306(1979), pp. 94-121.